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A THEORY OF HIGH FREQUENCY VIBRATIONS OF PIEZOELECTRIC CRYSTAL BARS

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Dedicated to Prof. Dr. Raymond D. Mindlin on his sixty-fifth birthday

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Abstract—This paper presents a higher order, linear theory of piezoelectric crystal bars as in the same spirit as that of Mindlin. First, a power series representation in aerial coordinates is employed for both the mechanical displacement and electric potential fields. Next, with the help of a variational theorem deduced from Hamilton's principle, together with these series, the theory is established consistently. A hierarchy of 1-dimensional approximate equations of motion, charge equations of electrostatics, initial and boundary conditions, strain–displacement and electric field–electric potential relations, and macroscopic constitutive equations constitutes the theory, and it governs all the types of motions of piezoelectric crystal bars of uniform cross-section. Further, special cases of interest are pointed out. The solutions of the initial mixed boundary-value problems defined by this theory are proven to be unique.

NOTATION

Throughout this paper we use Cartesian tensors in the Euclidean 3-space $\mathcal{E}$. Einstein's summation convention is implied for all repeated Latin and Greek indices, unless they are enclosed with parentheses. Latin indices range over 1–3, while Greek indices over 1 and 2. A comma followed by an index stands for partial differentiation and a star for prescribed quantities. A prime and a superposed dot are designated to denote partial differentiation with respect to the axial coordinate x_1 and to time t, respectively.

The list of notation follows:

L	length of bar
$\mathscr{A}$	cross-sectional area of bar
B, ∂B	entire volume of bar and its boundary surface
$\mathscr{C}$	a Jordan curve which bounds cross-section $\mathscr{A}$
t	time
x_i	Cartesian coordinates ($i = 1, \alpha$; $\alpha = 2, 3$)
$\mathbf{u}$, $\mathbf{d}$	mechanical displacement and electric displacement vectors
ϵ_{ij}	components of strain tensor
ε_{ijk}	permutation symbol
τ_{ij}	components of symmetric stress tensor
ρ	mass density
δ_{ij}	Kronecker's delta
U, K, H	internal energy, kinetic energy and electric enthalpy of bar
$\mathbf{e}$	electric field vector
φ	electric potential
$\mathbf{f}$, $\boldsymbol{\tau}$	body force and traction vectors
$\mathbf{n}$, $\boldsymbol{\nu}$	unit vectors normal to ∂B and to $\mathscr{C}$
σ	surface charge

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C_{ijkl}, C_{ijk}, C_{ij}	components of elastic stiffness, piezoelectric strain constants and dielectric permittivity
$T_{ij}^{(m,n)}$, $\ddot{A}_i^{(m,n)}$, $D_i^{(m,n)}$	components of stress, acceleration and electric displacement resultants defined in equation (3.4)
$F_i^{(m,n)}$, $T_i^{(m,n)}$ $D^{(m,n)}$	components of body force, traction and surface charge resultants defined in equation (3.4)
$I_{(m,n)}$	moment of inertia of higher order defined in equation (3.4)
$\epsilon_{ij}^{(m,n)}$, $e_i^{(m,n)}$	components of strain and electric field of order (m, n), defined in equation (3.2)
$\varphi^{(m,n)}$, $u_i^{(m,n)}$	electric potential and components of mechanical displacement of order (m, n)
S_φ, S_u	surface parts of ∂B, where the (t_i, φ) and the (u_i, σ) are prescribed
$(\dot{\ })$, $(\ )'$	$\frac{\partial}{\partial t}(\ )$, $\frac{\partial}{\partial x_1}(\ )$.

1. INTRODUCTION

Recent technological interest in piezoelectric crystals is well-known; a whole branch of industry is, in fact, devoted to the development of piezoelectric devices as a result of their extensive use for both military and non-military purposes. Consequently, a great deal of investigations have been carried out both experimentally and theoretically. The reader is referred to[1–6], and in particular, to a recent, excellent survey due to Mindlin[7] for the review of the earlier works on piezoelectric crystals.

Previously, Mindlin and his students have studied various types of oscillations of piezoelectric crystal plates, and their works have been recently elaborated by Tiersten[6] in a comprehensive monograph. More recently, Dökmeci[8] has formulated a generalized variational theorem for linear piezoelectricity, and then, he used it in order to construct a theory of piezoelectric crystal finite surfaces. Further, this theory is extended to include the effects of the elastic stiffness and inertia of the electrodes in crystal surfaces completely coated with electrodes on both faces[9]. A higher order theory of piezoelectric crystal bars is first presented in this paper. The theory governs the extensional, flexural and torsional oscillations of bars of uniform cross-section for both low and high frequencies.

Section 2 contains the fundamental equations of linear piezoelectricity and the variational theorem of [8]. This is followed by the 1-dimensional, approximate, macroscopic equations of piezoelectric bars in Section 3. First, the mechanical displacement and electric potential fields are represented by power series in terms of cross-sectional coordinates. Next, with the aid of the variational theorem and these series, we consistently establish a hierarchy of equations of motion, charge equations of electrostatics, initial and boundary conditions, strain–displacement and electric field–electric potential relations and appropriate constitutive equations. Then in Section 4, we enumerate the conditions to ensure the uniqueness for the solutions of the initial mixed boundary value problem described by the governing equations of piezoelectric crystal bars. In the last Section, special cases of interest are indicated and some conclusions concerning the theory are drawn.

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2. BASIC EQUATIONS OF LINEAR PIEZOELECTRICITY

A complete description of the linear theory of piezoelectricity is given, for instance, in [2], [6], [7] and [10]. The latter also includes the coupling between mechanical, electrical and thermal fields. For ease of reference, the following brief account is recorded.

In the Euclidean 3-space $\mathcal{E}$, let B stand for a regular region of space [11], occupied by the piezoelectric medium, with its closure $\bar{B}$ and boundary ∂B at time $t = t_0$, $\mathbf{n}$ for the unit outward vector normal to ∂B, and S_φ and S_u for the complementary regular subsurfaces of ∂B such that $S_\varphi \cup S_u = \partial B$ and $S_\varphi \cap S_u = 0$. Further, let $\bar{B} \times [t_0, t_1)$ be the domain of definitions for the functions of $(\mathbf{x}, t)$, where x_i being a fixed system of Cartesian coordinates and $t_1 > t_0$ may be infinity.

Now, we state the equations of local balance of momenta:

$$\tau_{ij,i} + f_j - \rho \ddot{u}_j = 0 \quad \text{on} \quad B \times [t_0, t_1) \tag{2.1}$$

$$\varepsilon_{ijk} \tau_{jk} = 0 \quad \text{on} \quad B \times [t_0, t_1). \tag{2.2}$$

Here, τ_{ij} is the components of symmetric stress tensor, $\mathbf{f}$ the body force vector per unit volume, $\mathbf{u}$ the displacement vector, ρ the mass density, and ε_{ijk} the usual permutation symbol. The stress vector $\boldsymbol{\tau}$ can be expressed:

$$\tau_j = n_i \tau_{ij}. \tag{2.3}$$

Maxwell's equations for the quasi-static electric field are given by

$$\mathbf{d}_{i,i} = 0 \quad \text{on} \quad B \times [t_0, t_1) \tag{2.4}$$

$$\mathbf{e}_i = -\varphi_{,i} \quad \text{on} \quad B \times [t_0, t_1) \tag{2.5}$$

in which $\mathbf{d}$ is the electric displacement vector, $\mathbf{e}$ the electric field vector, and φ the electric potential. The surface charge σ can be expressed in terms of the components of the electric displacement:

$$\sigma = n_i d_i. \tag{2.6}$$

The linear constitutive equations of uncoupled piezoelectricity are:

$$\tau_{ij} = C_{ijkl} \epsilon_{kl} - C_{kij} e_k \quad \text{on} \quad B \times [t_0, t_1) \tag{2.7}$$

$$d_i = C_{ijk} \epsilon_{jk} + C_{ij} e_j \quad \text{on} \quad B \times [t_0, t_1) \tag{2.8}$$

along with the symmetry conditions

$$C_{ijkl} = C_{jikl} = C_{klij} = C_{ijlk}, \quad C_{ijk} = C_{ikj}, \quad C_{ij} = C_{ji} \quad \text{on} \quad B \times [t_0, t_1) \tag{2.9}$$

where C_{ijkl}, C_{ijk} and C_{ij} denote the components of elastic stiffness ($e_i = \text{const.}$), piezoelectric strain constants and dielectric permittivity ($\epsilon_{ij} = \text{const.}$), respectively. The piezoelectric strain constants C_{ijk} play the coupling role between elastic and dielectric phenomena. The components of the strain tensor are defined through

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{on} \quad B \times [t_0, t_1). \tag{2.10}$$

The boundary conditions are written in a general form:

$$\tau_j^* - n_i \tau_{ij} = 0 \quad \text{on} \quad S_\tau \times [t_0, t_1), \quad \varphi_* - \varphi = 0 \quad \text{on} \quad S_\varphi \times [t_0, t_1) \tag{2.11}$$

$$u_i^* - u_i = 0 \quad \text{on} \quad S_u \times [t_0, t_1), \quad \sigma_* - n_i d_i = 0 \quad \text{on} \quad S_\sigma \times [t_0, t_1) \tag{2.12}$$

with S_τ, S_u and S_φ, S_σ being the complementary portions of ∂B. Here, the τ_i is prescribed on S_τ, the u_i on S_u, the σ on S_σ, and the φ on S_φ. For the sake of simplicity,

$$S_\tau = S_\varphi, \quad S_u = S_\sigma \tag{2.13}$$

is taken in the analysis.

Lastly, a set of the initial conditions, namely

$$\left.\begin{aligned} \dot{\mathbf{u}}(x_i, t_0) = \mathbf{v}^*(x_i), \quad \mathbf{u}(x_i, t_0) = \mathbf{w}^*(x_i) \\ \mathbf{e}(x_i, t_0) = \mathbf{e}^*(x_i) \end{aligned}\right\} \quad \text{on} \quad B(t_0) \tag{2.14}$$

completes the fundamental equations of the linear uncoupled theory of piezoelectricity. Here, the symbol $B(t)$ refers to B at time t. Furthermore, we should note that in the fundamental equations $\mathbf{u} \in C^{(1,2)}$, $\epsilon_{ij} \in C^{(0,0)} \tau_{ij} \in C^{(1,0)}$, $\mathbf{f} \in C^{(0,0)}$, $\mathbf{d} \in C^{(1,0)}$, $\mathbf{e} \in C^{(0,0)}$ and $\varphi \in C^{(1,0)}$ are assumed, where $C^{(m,n)}$ represents the functions with derivatives of order up to and including m and n with respect to x_i and t, respectively, provided that the functions and their derivatives exist and are continuous on $\bar{B} \times [t_0, t_1)$.

In closing this section, we recall the variational theorem [8, 12]:

$$\delta \mathscr{T}_{ii} = 0 \tag{2.15a}$$

with t_0 and t_1 denoting two arbitrary instants of time, and

$$\begin{aligned} \delta \mathscr{T}_{11} &= \int_{t_0}^{t_1} \mathrm{d}t \int_B [(\tau_{ij,i} + f_j - \rho \ddot{u}_j)\, \delta u_j + (d_{i,i})\, \delta\varphi]\, \mathrm{d}V \\ \delta \mathscr{T}_{22} &= \int_{t_0}^{t_1} \mathrm{d}t \int_{S_\varphi} [(\tau_j^* - n_i \tau_{ij})\, \delta u_j + (\varphi - \varphi_*)\, \delta\sigma]\, \mathrm{d}S \\ \delta \mathscr{T}_{33} &= \int_{t_0}^{t_1} \mathrm{d}t \int_{S_u} [(u_i - u_i^*)\, \delta\tau_i + (\sigma_* - n_i d_i)\, \delta\varphi]\, \mathrm{d}S. \end{aligned} \tag{2.15b}$$

This variational theorem is directly deduced from Hamilton's principle, and it leads to equations (2.1), (2.4), (2.11) and (2.12) as the appropriate Euler equations. The theorem, is, in fact, slightly different than the one due to the author. Here, by the suggestion of R. D. Mindlin, the traction together with electric potential (or displacement with charge) in lieu of the traction with surface charge (or displacement with potential) is prescribed; the former is the most commonly encountered in practice.

3. GOVERNING EQUATIONS OF PIEZOELECTRIC CRYSTAL BARS

In general, the direct method, the asymptotic method and the method of series expansion are the most suitable methods in order to establish continuum theories in 1- and 2-dimensions (see, e.g. [13–17] and references therein), of which the latter has been developed by Cauchy and Poisson, and it has been recapitulated by Mindlin[16, 18], who has extensively employed it in constructing beam and plate theories, and has shown its consistency [19]. This method of series expansion is used for the present study as well.

Electric potential and displacement fields

Consider a slender, cylindrical, piezoelectric crystal bar with no singularities of any type in $\mathcal{E}$ (Fig. 1). The bar is referred to a system of right-handed Cartesian coordinates x_i in this space. The coordinate axes are located at the centroid of the initial cross-section of the bar.

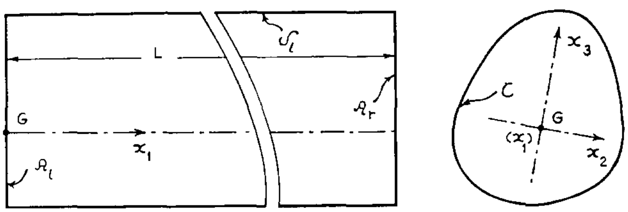

Fig. 1. Geometry of bar.

The x_1-axis is the centroidal bar axis; the x_2- and x_3-axes are chosen as the principal axes of cross-section. The mechanical displacement and electric potential of the bar are consistently approximated by the power series in aerial coordinates x_α (cf. [20, 21]), that is,

$$\mathbf{u}(x_i, t) = \sum_{m,n=0}^{M,N} x_2^m x_3^n \mathbf{u}^{(m,n)}(x_1, t) \tag{3.1a}$$

and

$$\varphi(x_i, t) = \sum_{m,n=0}^{M,N} x_2^m x_3^n \varphi^{(m,n)}(x_1, t). \tag{3.1b}$$

Here, it is assumed that $\mathbf{u}^{(m,n)}$ and $\varphi^{(m,n)}$ exist, and they are unknown *a priori* and independent functions to be determined.

Electric field and strain distributions

Substituting equation (3.1) into equations (2.5) and (2.10), we readily obtain the strain distributions:

$$\{\epsilon_{ij}(\mathbf{x}, t), e_i(\mathbf{x}, t)\} = \sum_{m,n=0}^{M,N} x_2^m x_3^n \{\epsilon_{ij}^{(m,n)}(x_1, t), e_i^{(m,n)}(x_1, t)\} \tag{3.2a}$$

where

$$\begin{aligned}
\epsilon_{\alpha\beta}^{(m,n)} &= \tfrac{1}{2}[(m+1)(\delta_{2\alpha} u_\beta^{(m+1,n)} + \delta_{2\beta} u_\alpha^{(m+1,n)}) + (n+1)(\delta_{3\alpha} u_\beta^{(m,n+1)} + \delta_{3\beta} u_\alpha^{(m,n+1)})] \\
\epsilon_{\alpha 1}^{(m,n)} &= \tfrac{1}{2}[u_\alpha'^{(m,n)} + (m+1)\,\delta_{2\alpha} u_1^{(m+1,n)} + (n+1)\,\delta_{3\alpha} u_1^{(m,n+1)}] \\
\epsilon_{11}^{(m,n)} &= u_1'^{(m,n)}
\end{aligned} \tag{3.2b}$$

and

$$e_\alpha^{(m,n)} = -[(m+1)\,\delta_{2\alpha}\varphi^{(m+1,n)} + (n+1)\,\delta_{3\alpha}\varphi^{(m,n+1)}],\ e_1^{(m,n)} = -\varphi'^{(m,n)} \tag{3.2c}$$

Equations of motion and of electrostatics

If the power series expansions (3.1) are inserted into the volume integral of equations (2.15), and then the integrations over the cross-sectional area of bar are carried out, we arrive at the equation:

$$\begin{aligned}
\delta\mathcal{T}_{11} = \int_{t_0}^{t_1} \mathrm{d}t \int_0^L \sum_{m,n=0}^{M,N} [(T_{1i}'^{(m,n)} - mT_{2i}^{(m-1,n)} - nT_{3i}^{(m,n-1)} + P_i^{(m,n)} - \rho\ddot{A}_i^{(m,n)})\,\delta u_i^{(m,n)} \\
+ (D_1'^{(m,n)} - mD_2^{(m-1,n)} - nD_3^{(m,n-1)} + D^{(m,n)})\,\delta\varphi^{(m,n)}]\,\mathrm{d}x_1
\end{aligned} \tag{3.3}$$

where L is the length of bar, and the quantities:

$$\begin{aligned}
&\{T_{ij}^{(m,n)}, D_i^{(m,n)}\} = \int_{\mathscr{A}} x_2^m x_3^n \{\tau_{ij}, d_i\}\,\mathrm{d}A \\
&\{T_i^{(m,n)}, D^{(m,n)}\} = \oint_{\mathscr{C}} x_2^m x_3^n \{\tau_{\alpha i}, d_\alpha\}\,\mathbf{v}_\alpha\,\mathrm{d}s \\
&P_i^{(m,n)} = F_i^{(m,n)} + T_i^{(m,n)}, \qquad F_i^{(m,n)} = \int_{\mathscr{A}} x_2^m x_3^n f_i\,\mathrm{d}A \\
&A_i^{(m,n)} = \sum_{p,q=0}^{M,N} I_{(m+p,n+q)} u_i^{(p,q)}, \qquad I_{(m,n)} = \int_{\mathscr{A}} x_2^m x_3^n\,\mathrm{d}A
\end{aligned} \tag{3.4}$$

are defined. $\mathscr{C}$ is a simply-connected Jordan curve which bounds $\mathscr{A}$; ds is the line element of $\mathscr{C}$ and $\mathbf{v}$ the unit exterior normal vector on $\mathscr{C}$. It is noteworthy that

$$A = I_{(0,0)}, \qquad I_{(0,1)} = I_{(1,0)} = I_{(1,1)} = 0, \qquad I_{(2,0)} = I_{x_3x_3}, \qquad I_{(0,2)} = I_{x_2x_2} \tag{3.5}$$

in the notation of equations (3.4), since the principal axes x_α are located at the centroid of cross-section.

The variations $\delta u_i^{(m,n)}$ and $\delta\varphi^{(m,n)}$ in equation (3.3) are independently and arbitrarily taken. Hence we obtain the hierarchy of 1-dimensional, approximate equations of motion and of electrostatics as follows:

$$T_{1i}^{\prime(m,n)} - mT_{2i}^{(m-1,n)} - nT_{3i}^{(m,n-1)} + P_i^{(m,n)} - \rho\ddot{A}_i^{(m,n)} = 0 \quad \text{on} \quad L \times [t_0, t_1) \tag{3.6}$$

$$D_1^{\prime(m,n)} - mD_2^{(m-1,n)} - nD_3^{(m,n-1)} + D^{(m,n)} = 0 \quad \text{on} \quad L \times [t_0, t_1) \tag{3.7}$$

Natural boundary conditions

The surface integrals in equation (2.15) are evaluated using the series expansion (3.1) as in the volume integral, and they are found to be

$$\left.\begin{aligned} &\mathbf{u}^{(m,n)} - \mathbf{u}^{*(m,n)} = 0, \qquad D^{*(m,n)} - D^{(m,n)} = 0 \quad \text{on} \quad S_d \times [t_0, t_1)\\ &T_i^{*(m,n)} - T_i^{(m,n)} = 0, \qquad \varphi^{(m,n)} - \varphi^{*(m,n)} = 0 \quad \text{on} \quad S_t \times [t_0, t_1)\end{aligned}\right\} \tag{3.8}$$

and

$$\left.\begin{aligned} &M_i^{*(m,n)} - T_{1i}^{(m,n)} = 0, \qquad \varphi^{(m,n)} - \psi^{*(m,n)} = 0 \quad \text{on} \quad \mathscr{A}_r \times [t_0, t_1)\\ &\mathbf{u}^{(m,n)} - \mathbf{v}^{*(m,n)} = 0, \qquad \Theta^{*(m,n)} + D_1^{(m,n)} = 0 \quad \text{on} \quad \mathscr{A}_l \times [t_0, t_1)\end{aligned}\right\} \tag{3.9}$$

for independent and arbitrary variations indicated. Here, $v^{*(m,n)}$ and $\psi^{(m,n)}$ are given functions. Also we have introduced:

$$\{M_i^{*(m,n)}, \Theta^{*(m,n)}\} = \int_{\mathscr{A}} x_2^m x_3^n \{\tau_i^*, \sigma^*\}\, dA, \quad \{T_i^{*(m,n)}, D^{*(m,n)}\} = \oint_{\mathscr{C}} x_2^m x_3^n (\tau_i^*, \sigma^*)\, ds. \tag{3.10}$$

and considered:

$$S_l \cup S_e = \partial B, \quad S_l = S_d \cup S_t, \quad S_u = S_d \cup \mathscr{A}_l, \quad S_\varphi = S_t \cup \mathscr{A}_r, \quad S_e = \mathscr{A}_l \cup \mathscr{A}_r. \tag{3.11}$$

In this equation, S_l, S_e, $\mathscr{A}_l$ and $\mathscr{A}_r$ are the lateral, edge, left face and right face boundary surfaces of bar, respectively. The $\mathbf{u}$ and σ are prescribed on the portion S_d of S_l and on $\mathscr{A}_l$, while the τ and φ are prescribed on the remaining portion S_t of S_l and on $\mathscr{A}_r$. In the coordinate system used, the exterior normal vector $\mathbf{n}$ takes the values $n_1 = 1$ on $\mathscr{A}_r$ and $n_1 = -1$ on $\mathscr{A}_l$.

Constitutive equations

With the aid of equations (2.7–2.9) and equations (3.2), the linear macroscopic constitutive equations of piezoelectric crystal bars are obtained:

$$T_{ij}^{(m,n)}(x_1, t) = \sum_{p,q}^{M,N} I_{(m+p,n+q)}(C_{ijkl}\,\epsilon_{kl}^{(p,q)} - C_{kij}\,e_k^{(p,q)}) \quad \text{on} \quad L \times [t_0, t_1) \tag{3.12}$$

$$D_i^{(m,n)}(x_1, t) = \sum_{p,q}^{M,N} I_{(m+p,n+q)}(C_{ijk}\,\epsilon_{jk}^{(p,q)} + C_{ij}\,e_j^{(p,q)}) \quad \text{on} \quad L \times [t_0, t_1) \tag{3.13}$$

Initial conditions

Now, we make use of equations (2.5), (2.14) and (3.1), and read the initial conditions, namely

$$\mathbf{u}^{(m,n)}(x_1, t_0) = \boldsymbol{\alpha}^{*(m,n)}(x_1), \quad \dot{\mathbf{u}}^{(m,n)}(x_1, t_0) = \boldsymbol{\beta}^{*(m,n)}(x_1); \quad \varphi^{(m,n)}(x_1, t_0) = \gamma^{*(m,n)}(x_1) \quad \text{on} \quad L(t_0) \tag{3.14}$$

where $\boldsymbol{\alpha}^*$, $\boldsymbol{\beta}^*$ and γ^* are given functions.

Linear theory of piezoelectric crystal bars

The union of the electric potential and displacement fields (3.1), the electric field and strain distributions (3.2), the equations of motion (3.6) and of electrostatics (3.7), the natural boundary conditions (3.8–3.9), the constitutive relations (3.12–3.13), and the initial conditions (3.14) constitutes the governing equations of the linear theory of piezoelectric crystal bars.

4. UNIQUENESS OF SOLUTION

In this section, we establish a theorem of uniqueness for the solutions of the governing equations formulated previously. Of the several arguments used to construct uniqueness theorems of elastodynamics, it is the classical energy argument due to Neumann[22, 23], upon which the present theorem of uniqueness is based. To begin with, we state the following theorem.

Theorem. There exists, at most, one solution to the initial mixed boundary-value problem defined by the aforementioned governing equations for the finite bar space $B + \partial B$ with boundary $\partial B(\partial B = S_u \cup S_\varphi, S_u \cap S_\varphi = 0)$ in $\mathcal{E}$.

To prove the theorem, we consider, as usual, two sets of solutions. Since all the governing equations are linear, the difference set of the two solutions is a solution of the homogeneous governing equations. Consequently, it is enough to show that these homogeneous equations have only the trivial solution.

Now, we express the rates of the kinetic energy K, internal energy U, and electric enthalpy H of bar as

$$\dot{K} = \int_B \rho \ddot{u}_i \dot{u}_i \, dv, \qquad \dot{H} = \int_B (\tau_{ij} \dot{\epsilon}_{ij} - d_i \dot{e}_i) \, dv, \qquad \dot{U} = \dot{H} + \int_B (e_i d_i)^{\cdot} \, dv. \tag{4.1}$$

Here, we should note that both the kinetic and internal energies are positive-definite, and this is enough to yield a unique solution. On using equations (3.1), (3.2) and (3.4), we obtain these rates in the form:

$$\begin{aligned}\dot{K} &= \int_0^L \sum_{m,n=0}^{M,N} \rho \ddot{A}_i^{(m,n)} \dot{u}_i^{(m,n)} \, dx_1 \\ \dot{U} &= \int_0^L \sum_{m,n=0}^{M,N} [T_{i1}^{(m,n)} \dot{u}_i'^{(m,n)} + (m T_{i2}^{(m-1,n)} + n T_{i3}^{(m,n-1)}) \dot{u}_i^{(m,n)} - \dot{D}_1^{(m,n)} \varphi'^{(m,n)} \\ &\qquad - (m \dot{D}_2^{(m-1,n)} + n \dot{D}_3^{(m,n-1)}) \varphi^{(m,n)}] \, dx_1.\end{aligned} \tag{4.2}$$

With the help of equation (3.6), we form the homogeneous equation:

$$\int_{t_0}^{t_1} dt \int_0^L \sum_{m,n=0}^{M,N} (T_{1i}'^{(m,n)} - m T_{2i}^{(m-1,n)} - n T_{3i}^{(m,n-1)} + T_i^{(m,n)} - \rho \ddot{A}_i^{(m,n)}) \dot{u}_i^{(m,n)} \, dx_1 = 0. \tag{4.3}$$

In view of the total energy rates (4.2), this equation may be written:

$$\begin{aligned}\int_{t_0}^{t_1} dt \Big\{ -(\dot{U} + \dot{K}) + \int_0^L \sum_{m,n=0}^{M,N} [T_{i1}^{(m,n)} \dot{u}_i'^{(m,n)} + (T_i^{(m,n)} + T_{i1}'^{(m,n)}) \dot{u}_i^{(m,n)} \\ - \dot{D}_1^{(m,n)} \varphi'^{(m,n)} - (m \dot{D}_2^{(m-1,n)} + n \dot{D}_3^{(m,n-1)}) \varphi^{(m,n)}] \Big\} \, dx_1 = 0\end{aligned} \tag{4.4}$$

which, after performing the pertinent integrations and using equation (3.7), becomes:

$$U(t_1) + K(t_1) = U(t_0) + K(t_0) + \sum_{m,n=0}^{M,N} \int_{t_0}^{t_1} \mathrm{d}t \left\{ \int_0^L [T_i^{(m,n)} \dot{u}_i^{(m,n)} - (\dot{D}^{(m,n)} \varphi^{(m,n)}] \, \mathrm{d}x_1 + (T_{i1}^{(m,n)} \dot{u}_i^{(m,n)} - \dot{D}_1^{(m,n)} \varphi^{(m,n)}) \Big|_{x_1=0}^{L} \right\}. \quad (4.5)$$

To ensure a unique solution, the terms under the integral sign in this equation should vanish. Evidently, the conditions (3.8–3.9) and (3.14) as well as to specify any member of each terms under the integral sign assure the uniqueness. In fact, the vanishing integrals in equation (4.5) together with the positive-definiteness of the U and K lead to

$$U(t_1) + K(t_1) = U(t_0) + K(t_0)$$

which implies a trivial solution for the displacement and electric potential fields, and completes the proof of the theorem.

5. DISCUSSION

In this study a systematic development of higher order, 1-dimensional, linear theory of piezoelectric crystal bars is presented. Also, a theorem of uniqueness in this theory is established and the sufficient conditions for a unique solution are obtained. On the basis of the 3-dimensional field equations of piezoelectricity, the theory is consistently formulated by the use of a variational theorem together with a method of series expansion, and it governs all the types of motions of nonpolar bars of constant cross-section, for low as well as high frequencies.

Clearly, the variational theorem (2.15) well serves only in constructing the macroscopic equations of motion, charge equations of electrostatics and natural boundary conditions of piezoelectric crystal bars. A complete variational derivation of the governing equations can be possible by means of a variational theorem developed recently by Dökmeci[24]. This theorem may generate all the governing equations of linear piezoelectricity, including the most general form of the boundary conditions as well as the initial and jump conditions (cf. [25] in classical elastodynamics).

The method of series expansion adopted here might be successfully employed for other field quantities such as electric potential in lieu of electric displacements (this has been illustrated for piezoelectric crystal plates in [6]) as a starting point. Nevertheless, the present choice, i.e. mechanical displacement and electric potential, is, in general, more tractable than any others in deriving approximate continuum theories. Further, by applying this method, the theory can be extended, in a straightforward manner, to obtain that of crystal bars, coated with electrodes (cf. [9]).

When the terms of higher than zero and one are discarded, we respectively obtain the Bernoulli and Timoshenko theories of piezoelectric crystal bars (cf. [17, 19]). Moreover, on account of the macroscopic constitutive equations (3.12, 3.13), the equations of motion (3.6) are coupled to the charge equations of electrostatics (3.7). If, however, the terms involved with piezoelectricity are dropped out, the theory recovers the isothermal linear beam theories due to Mindlin[19], Warner[26], Bleustein and Stanley[27], and Dökmeci[17].

Lastly, a large class of applications based on this theory will form the subject of forthcoming studies.

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Абстракт — В работе представляется линейная теория высшего порядка для пьезоэлектрических кристаллических брусов, в таком же самом смысле как по теории Миндлина. Впервые, применяется воспроизведение в степенных рядах для пространственных координат, как для полей механических перемещений, так и для полей электрического потенциала. Затем с помощью вариационной теоремы, выведенной из принципа Гамильтона, вместе с этими рядами, создавается логично теория. Иерархия одномерных приближенных уравнений движения, уравнений заряда электростатики, начальных и граничных условий, соотношений между относительной деформацией и смещением и между электрическим полем и электрическим потенциалом, и макроскопических уравнений состояния учреждает теорию и, далее, определяет все типы движений для пьезоэлектрических кристаллических брусов, равномерного поперечного сечения. Кроме того, обращается внимание на специальные интересные случаи. Доказывается, что решения задач для начальных смешанных краевых задач единствены.